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ON A CERTAIN CRITERION OF SHYNESS FOR SUBSETS IN THE PRODUCT OF UNIMODULAR POLISH GROUPS THAT ARE NOT COMPACT

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Abstract

We give an extension of the notion of generators of shy sets in Polish topological vector spaces [8] to all Polish groups. By a new approach supported on the technique of generators of shy sets, we extend Randal Dougherty criterion of shyness for subsets (cf. [3], Proposition 12, pp. 87) in the product of unimodular Polish groups that are not compact

1. Introduction

A suitable extension of the property of being of Haar measure zero in abelian Polish groups [2] to all non-abelian Polish groups is given in Topsøe and Hoffmann-Jørgens [12] and Mycielski [7] as follows: A Borel set A in a Polish group G is called *shy,* if there exists a Borel measure μ over G such that $\mu(F) > 0$ for any compact $F \subset G$ and every two sided translation of A is of μ- measure zero; similarly, a Borel set B in a Polish

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topological group G is called *left* (or *right*)-shy, iff there exists a Borel measure λ over G such that every left (or right) translation of B is of λ- measure zero and $\lambda(F) > 0$ for any compact $F \subset G$. The measure μ is said to be a transverse to A, and a measure λ is said to be a left (or right) transverse to B. A subset of G is called (left or right) shy, if it is subset of a Borel (left or right) shy set. Also, we preserve notions of transverse measures for subsets of Borel (left or right) shy sets.

It can be mentioned especially, the deep result of Topsøe and Hoffmann-Jørgens [12] and Mycielski [7] asserted *that the class of (left or right) shy sets in a Polish group is an* σ- *ideal*. Starting from this point, there are many observations devoted to construct of criterions of shyness for subsets in various Polish groups. Among of them, the result of Dougherty [3] can be mentioned especially, which asserts that *all compact subsets of a non-locally compact Polish group that has an invariant metric are shy*. For example, this result follows that, if G is a Polish group with a two sided invariant metric that is σ- compact but not compact, (e.g., $\mathbb{R}$ or $\mathbb{Z}$) and X is the set of eventually constant sequences in $G^{\mathbb{N}}$, then X is shy. Indeed, it is easy to check that X is σ- compact, and hence Dougherty's theorem implies that it is shy. A question whether compact sets are (left or right) shy sets in a non-locally compact Polish group, still remains open (see, [11], Problem 3, pp. 43).

The purpose of the present paper is to extend Dougherty criterion of shyness for subsets (see [3], Proposition 12) in the product of a countable family of unimodular Polish groups that are not compact.

The paper is organized as follows.

In Section 2, we give an extension of the notion of generators of shy sets introduced for Polish topological vector spaces [8] to all Polish groups. We prove that the class of generators of shy sets in an abelian Polish group G is non-empty, if G contains any uncountable locally compact Hausdorff topological subgroup. This result (see, Theorem 2.1) extends one result obtained for Polish topological vector spaces in [8] (see, Theorem 2.1, pp. 238). In addition, we show that, if G has an invariant metric, then every generator of shy sets on G is non-σ- finite.

In Section 3, we construct left (or right)-invariant generators of shy sets on the product of locally compact Hausdorff topological groups. Analogously, we give an example of a two sided invariant generator of shy sets in the product of unimodular Polish groups. For every compact set K in the product of a countable family of non-compact unimodular Polish groups, we construct a two sided invariant generator of shy sets μ such that $\mu(K) = 0$.

In Section 4, we present the main result of our paper and construct such an example of a shy set in the product of a countable family of unimodular Polish groups (that are not compact), which is not described by Dougherty above mentioned criterion of shyness for subsets.

2. On Generators of Shy Sets on Polish Groups

Let G be a Polish group, by which, we mean a group with a complete metric for which the transformation (from $G \times G$ onto G), which sends (x, y) into $x^{-1}y$ is continuous. Let $B(G)$ denotes the σ-algebra of Borel subsets of G, and μ be a nonzero nonnegative measure defined on $B(G)$.

As the notes of Christensen [2] (pp.119), in the non-locally compact abelian Polish group, there is no σ-finite (equivalently, probability) measure μ such that S being shy is equivalent to $\mu(S) = 0$. Slightly more can be said that any σ-finite Borel measure μ must assign 0 to a prevalent set of points. On these ground, we introduce the following definitions.

Definition 2.1. A Borel measure μ on G is called a *generator of shy sets in* G, if

$$(\forall X)(\overline{\mu}(X) = 0 \rightarrow X \in S(G)),$$

where $\overline{\mu}$ denotes a usual completion of the Borel measure μ, and $S(G)$ denotes a class of all shy sets.

Definition 2.2. A Borel measure μ on G is called a *generator of left* (or *right*) *shy sets in* G, if

$$(\forall X)(\overline{\mu}(X) = 0 \to X \in \mathcal{LS}(G)(\text{or } \mathcal{RS}(G)),$$

where $\mathcal{LS}(G)$ and $\mathcal{RS}(G)$ denote classes of all left and right shy sets in G, respectively.

Definition 2.3. A Borel measure μ on G is called *quasi-finite*, if there exists a compact set $U \subseteq G$ for which $0 < \mu(U) < \infty$.

Definition 2.4. A Borel measure μ on G is called *semi-finite*, if for X with $\mu(X) > 0$, there exists a compact subset $F \subseteq X$ for which $0 < \mu(F) < \infty$.

Definition 2.5. A Borel measure on G is called *two sided quasi-invariant*, if μ and its every two sided transformation $g\mu f(g, f \in G)$ defined by

$$(\forall X)(X \in B(G) \to g\mu f(X) = \mu(gXf))$$

are equivalent.

Definition 2.6. A Borel measure on G is called *left quasi-invariant*, if μ and its every left transformations $g\mu(g \in G)$ defined by

$$(\forall X)(X \in B(G) \to g\mu(X) = \mu(gX))$$

are equivalent.

One can introduce similarly a notion of right quasi-invariant Borel measure on G.

Definition 2.7. A Borel measure μ on G is called *left invariant*, if

$$(\forall X)(\forall g)(X \in B(G) \,\&\, g \in G \to \mu(gX) = \mu(X)).$$

Definition 2.8. A Borel measure μ on G is called *right invariant*, if

$$(\forall X)(\forall g)(X \in B(G) \,\&\, g \in G \to \mu(Xg) = \mu(X)).$$

Definition 2.9. A Borel measure μ on G is called *two sided invariant*, if

$$(\forall X)(\forall g, f)(X \in B(G) \,\&\, g, f \in G \to \mu(gXf) = \mu(X)).$$

Definition 2.10. Let K be the class of measures on G. We say that a measure $\mu \in K$ has the property of uniqueness in the class K, if μ and λ are equivalent for every $\lambda \in K$.

There naturally arise the following questions.

Question 2.1. Let G be a Polish group. Whether there exists a generator of shy sets in G ?

Question 2.2. Let G be a Polish group and let a class of generators of shy sets in G is non-empty. Whether there exists a generator of shy sets with the property of uniqueness in the entire class of generators ?

As we know these questions still remain open.

The next assertion gives a positive answer to the Question 2.1 for such an abelian Polish group, which contains any uncountable locally-compact Hausdorff topological subgroup.

Theorem 2.1. *Let* $(G, +)$ *be an abelian Polish group which admits the following representation* $G = G_0 + G_1$, *where* G_0 *is an uncountable locally-compact Hausdorff topological group, and* G_1 *is such a group that* $G_0 \cap G_1 = \{\mathbf{0}\}$, *where* $\{\mathbf{0}\}$ *denotes the zero of the group* $(G, +)$. *Then, there exists a semi-finite inner regular invariant generator* λ *of shy sets in G. The generator* λ *is non*-σ-*finite, iff the* G_1 *is uncountable.*

Proof. Let μ be a Haar measure on G_0. For every $X \in B(G)$, we set

$$\lambda(X) = \sum_{g_1 \in G_1} \mu(X + g_1 \cap G_0).$$

Let us show that λ is a measure. Let $(X_k)_{k \in \mathbb{N}}$ be a family of pairwise disjoint Borel sets in G. Then we get

$$\lambda(\cup_{k \in \mathbb{N}} X_k) = \sum_{g_1 \in G_1} \mu((\cup_{k \in \mathbb{N}} X_k) + g_1 \cap G_0)$$

$$= \sum_{g_1 \in G_1} \mu(\cup_{k \in \mathbb{N}} (X_k + g_1) \cap G_0)$$

$$= \sum_{g_1 \in G_1} \mu(\cup_{k \in \mathbb{N}} (X_k + g_1 \cap G_0))$$

$$= \sum_{g_1 \in G_1} \sum_{k \in \mathbb{N}} \mu((X_k + g_1 \cap G_0))$$

$$= \sum_{k \in \mathbb{N}} \sum_{g_1 \in G_1} \mu((X_k + g_1 \cap G_0))$$

$$= \sum_{k \in \mathbb{N}} \lambda(X_k).$$

Let us show that the measure λ is quasi-finite.

Since μ is a regular Borel measure on G_0, there exists a compact set $F \subset G_0$ such that $0 < \mu(F) < \infty$. We have

$$\lambda(F) = \sum_{g_1 \in G_1} \mu(F + g_1 \cap G_0) = \mu(F \cap G_0) = \mu(F).$$

The measure λ is invariant. Indeed, for $f \in G$, there exists $f_0 \in G_0$ and $f_1 \in G_1$ such that $f = f_0 + f_1$. For $X \in B(G)$, we have

$$\lambda(X + f) = \sum_{g_1 \in G_1} \mu((X + f) + g_1 \cap G_0) = \sum_{g_1 \in G_1} \mu(X + f_1 + g_1 + f_0 \cap G_0)$$

$$= \sum_{g_1 \in G_1} \mu(X + f_1 + g_1 \cap G_0) = \sum_{f_1 + g_1 \in G_1 + f_1} \mu(X + f_1 + g_1 \cap G_0)$$

$$= \sum_{f_1 + g_1 \in G_1} \mu(X + f_1 + g_1 \cap G_0) = \sum_{h_1 \in G_1} \mu(X + h_1 \cap G_0) = \lambda(X).$$

Now, let us show that λ is a generator.

Let S be a subset of G with $\overline{\lambda}(S) = 0$. Since $\overline{\lambda}$ is a completion of λ, there exists a Borel set S' for which $S \subseteq S'$ and $\lambda(S') = 0$. Taking in to account that F is a compact set in G with $0 < \lambda(F) < \infty$, and applying a simple consequence of the invariance of the measure λ stating that $\lambda(S' + v) = 0$ for $v \in G$, we deduce that λ is transverse to a Borel set S'.

This means that S' is a Borel shy set. Finally, since S is a subset of a Borel shy set S'. We conclude that S is a shy set. One can observe that the generator λ is non-σ-finite iff card $(G_1) \geq \omega$.

Let us show that the generator λ is inner regular. For this, we are to show that

$(\forall X)(\forall \epsilon)(0 < \lambda(X) < \infty \,\&\, \epsilon > 0 \to (\exists F_\epsilon)(F_\epsilon \subseteq X \,\&\, F_\epsilon$ is compact $\&$ $\lambda(X \setminus F_\epsilon) < \epsilon))$.

Since, $0 < \lambda(X) < \infty$, there exists $I_0 \subset G_1$ such that card $(I_0) \leq \aleph_0$ and

$$0 < \lambda(X) = \sum_{g \in I_0} \mu(X + g \cap G_0) < \infty.$$

We set $I_0 = (g_m)_{m \in N}$. Let n_0 be a natural number such that

$$\sum_{1 \leq m \leq n_0} \mu(X + g_m \cap G_0) > \lambda(X) - \frac{\epsilon}{2}.$$

For $\in > 0$, there exists a compact set F_m such that $F_m \subseteq X + g_m$ and

$$\mu((X + g_m) \setminus F_m) < \frac{\epsilon}{2^{m+1}},$$

for $1 \leq m \leq n_0$.

We set $F_\epsilon = \cup_{1 \leq k \leq n_0}(F_k - g_k)$. It is obvious that F_ϵ is compact in V.

Finally, we get

$$\lambda(X \setminus \cup_{1 \leq k \leq n_0}(F_k - g_k)) = \sum_{g \in I_0} \mu((X \setminus \cup_{1 \leq k \leq n_0}(F_k - g_k)) + g \cap G_0)$$

$$= \sum_{m \in N} \mu(((X \setminus \cup_{1 \leq k \leq n_0}(F_k - g_k)) + g_m) \cap G_0)$$

$$= \sum_{1 \leq m \leq n_0} \mu(((X \setminus \cup_{1 \leq k \leq n_0}(F_k - g_k)) + g_m) \cap G_0)$$

$$+ \sum_{m>n_0} \mu(((X \setminus \cup_{1\le k\le n_0}(F_k - g_k)) + g_m) \cap G_0)$$

$$\le \sum_{1\le m\le n_0} \mu(((X \setminus (F_m - g_m)) + g_m) \cap G_0)$$

$$+ \sum_{m>n_0} \mu(X + g_m \cap G_0)$$

$$\le \sum_{1\le m\le n_0} \mu(((X + g_m) \setminus F_m) \cap G_0)$$

$$+ \sum_{m>n_0} \mu(X + g_m \cap G_0)$$

$$\le \sum_{1\le m\le n_0} \frac{\epsilon}{2^{m+1}} + \frac{\epsilon}{2}$$

$$\le \sum_{m\in N} \frac{\epsilon}{2^{m+1}} + \frac{\epsilon}{2} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Let us show that λ is semi-finite. Indeed, if $\lambda(X) > 0$, then there exists $g_0 \in G_1$ such that $0 < \mu(X + g_0 \cap G_0) < \infty$. Using the property of inner regularity of μ, we deduce that there exists a compact set $F \subseteq X$ with $0 < \lambda(F) < \infty$. The latter relation means that λ is semi-finite. □

Theorem 2.2. *Let G be a non-locally compact abelian Polish group with an invariant metric, which contains any uncountable locally compact Hausdorff topological group. Then the class of generators of shy sets in G is non-empty and each its element is non-σ-finite.*

Proof. By Theorem 2.1, we claim that the class of generators of shy sets in G is non-empty. Now assume that λ be any σ-finite element of this class. Then, there exists a countable family of compact sets $\{K_n : n \in N\}$ such that

$$\lambda(G \setminus \cup_{n\in N} K_n) = 0.$$

By Dougherty's result (see, [3], Proposition 12, pp.87), we claim that every compact set in G is shy. By using Mycielski well known result (cf. [7], Theorem 3, pp.32), the set $\cup_{n\in\mathbb{N}} K_n$ also is shy in G. By definition of a generator of shy sets in G, we claim that the set $G \setminus \cup_{n \in N} K_n$ being a set of λ- measure zero, is shy in G. Using again Mycielski above mentioned result, we deduce that G is shy, which is a required contradiction because there does not exist a Borel measure on G, which is transverse to the set G. □

The following corollary is immediate consequence of Theorems 2.1-2.2.

Corollary 2.1. *There always exists a semi-finite inner regular generator of shy sets in the Polish topological vector space* G, *and no any generator has a property of uniqueness*, *if* $dim(G) > 1$. *In addition, if G is infinite-dimensional*, *then every generator of shy sets in G is non*-σ- *finite.*

Remark 2.1. The result of Corollary 2.1 has been obtained in [8] (see, Theorem 2.1, pp. 238; Theorem 2.2, pp. 241).

By the scheme considered in the proof of Theorem 2.2, one can prove the following.

Proposition 2.1. *Every generator of shy sets in a non-locally compact Polish group G with an invariant metric is non*-σ- *finite.*

Theorem 2.3. *Every quasi-finite two sided quasi-invariant Borel measure* μ *defined on a Polish group G is a generator of shy sets in G.*

Proof. Let $\overline{\mu}(S) = 0$ for $S \subset G$. Since $\overline{\mu}(S) = 0$, there exists a Borel set S' for which $S \subseteq S'$ and $\mu(S') = 0$. By using the property of two sided quasi-invariance of the Borel measure μ, we have

$$(\forall f,\ g)(f,\ g \in G \rightarrow \mu(fS'g) = 0).$$

This relation together with the property of quasi-finiteness of μ implies that the Borel measure μ is transverse to the Borel set S' and therefore, S' is a Borel shy set. S is a shy set as a subset of the Borel shy set S'. □

Corollary 2.2. *Since every two sided invariant Borel measure is at the same time two sided quasi-invariant, we deduce that every quasi-finite two-sided invariant Borel measure on G is a generator of shy sets in G.*

3. On Generators of Shy Sets on the Product of Locally Compact Hausdorff Topological Groups

We begin this section by a certain observation belonging to Baker [1].

Let $(X_i, M_i, \mu_i)_{i\in\mathbb{N}}$ be a sequence of measure spaces and for each $i(i \in \mathbb{N})$, let ρ_i be a metric on the set X_i. Assume that the following conditions are satisfied.

(i) Each (X_i, ρ_i) is a locally compact metric space.

(ii) Each M_i contains the family, $\mathcal{B}(X_i)$, of Borel subsets of X_i and μ_i is a regular Borel measure on X_i (cf. [10], Definition 2.15).

(iii) For all i, and for every $\delta > 0$, there exists a sequence (A_j) of Borel subsets of X_i such that $d_i(A_j) < \delta$ and $X_i = \cup_{j=1}^{\infty} A_j$, where $d_i(A_j)$ is the diameter of A_j in X_i.

Define $X = \prod_{i=1}^{\infty} X_i$ and let equip X with the product topology. Let denote by $\mathcal{R}$, the family of all rectangles $R \subseteq X$ of the form $R = \prod_{i=1}^{\infty} R_i$, $R_i \in \mathcal{B}(X_i)$, and

$$0 \leq \prod_{i=1}^{\infty} \mu_i(R_i) := \lim_{n\to\infty} \prod_{i=1}^{n} \mu_i(R_i) < +\infty.$$

For $R \in \mathcal{R}$, we define $\tau(R) = \prod_{i=1}^{\infty} \mu_i(R_i)$.

Let τ^* be the set function on the powerset $\mathcal{P}(X)$ defined by

$$\tau^*(E) = \inf\{\sum \tau(R_j) : R_j \in \mathcal{R} \ \&\ E \subseteq \cup R_j\},$$

for $E \subseteq X$.

Also, let use the convention that $0 \cdot +\infty = +\infty \cdot 0 = 0$, $+\infty \cdot +\infty = +\infty$, and that the infimum taken over the empty set has the value $+\infty$.

Lemma 3.1 ([1], **Theorem I**, **pp. 2579**). *The set function* τ^* *is an outer measure on X. Let M be the* σ-*algebra of subsets of X which are measurable with respect to* τ^*, *and let* μ *be the measure on X obtained by restricting* τ^* *to* $\mathcal{B}(X)$. *Then, for all* $R = \prod_{i=1}^{\infty} R_i \in \mathcal{R}$, *we have* $\mu(R) = \prod_{i=1}^{\infty} \mu_i(R_i)$. *If each space* X_i *contains disjoint subsets* A_i, B_i *such that* $\mu_i(A_i) = \mu_i(B_i) = 1$, *then the measure* μ *is not* σ-*finite. Finally, assume that each* (X_i, ρ_i) *is an* M_i-*measurable group. If each* μ_i *is left-invariant measure on* M_i, *then* μ *is left-invariant Borel measure on X. Similarly, if each* μ_i *is right-invariant measure on* M_i, *then* μ *is right-invariant Borel measure on X.*

The following lemma is an immediate consequence of Lemma 3.1.

Lemma 3.2. *Under conditions of Lemma* 3.1, *if each* X_i *is a Polish group and* μ_i *is a two sided invariant Borel measure on* $X_i (i \in \mathbb{N})$, *then* μ *is a two sided invariant Borel measure on X.*

Proof. By the definition of the measure μ, for $g = (g_i)_{i \in \mathbb{N}}$, $f = (f_i)_{i \in \mathbb{N}} \in X$, and for $E \in \mathcal{B}(X)$, we get

$$\begin{aligned}
\mu(gEf) &= \inf\{\sum \tau(R_j) : R_j \in \mathcal{R} \ \&\ gEf \subseteq \cup R_j\} \\
&= \inf\{\sum \tau(R_j) : R_j \in \mathcal{R} \ \&\ E \subseteq \cup g^{-1} R_j f^{-1}\} \\
&= \inf\{\sum \tau(g^{-1} R_j f^{-1}) : g^{-1} R_j f^{-1} \in \mathcal{R} \ \&\ E \subseteq \cup g^{-1} R_j f^{-1}\} \\
&= \inf\{\sum \tau(R_j) : R_j \in \mathcal{R} \ \&\ E \subseteq \cup R_j\} = \mu(E). \qquad \square
\end{aligned}$$

Recall that a locally compact Hausdorff topological group G, whose left Haar measure coincides with its right Haar measure is called a unimodular group. Clearly, every abelian locally compact σ-compact Hausdorff topological groups are unimodular. There exist an examples of unimodular groups that are not abelian (see, for example, [6]). The definition of a unimodular group follows that its every Haar measure is two sided invariant. Indeed, if λ is a left Haar measure, then it must be its right Haar measure. Correspondingly, for every two elements $f, g \in G$, and for every Borel subset E in G, we get $\mu(fEg) = \mu(fE) = \mu(E)$.

Theorem 3.1. *For $i \in \mathbb{N}$, let G_i be a unimodular Polish group, that is, not compact and let μ_i be a Haar measure on G_i. Then the Borel measure μ defined by Lemma* 3.1 *is a two sided invariant generator of shy sets on the product-group $G = \prod_{i \in \mathbb{N}} G_i$.*

Proof. By Lemma 3.2, we claim that the Borel measure μ is a two-sided invariant Borel measure on G. An application of Corollary 2.2 ends the proof of Theorem 3.1. □

By Corollary 2.2 and Lemma 3.1, we can prove the following assertion.

Theorem 3.2. *For $i \in \mathbb{N}$, let G_i be a locally compact Hausdorff topological group, that is, not compact and let μ_i be a left (or right) Haar measure on G_i. Then the Borel measure μ defined by Lemma* 3.1 *is a left* (*or right*) *invariant generator of left* (*or right*) *shy sets on the product-group $G = \prod_{i \in \mathbb{N}} G_i$.*

Lemma 3.3. *Let G_1 and G_2 be two Polish groups. Then X is* (*left or right*) *shy in G_1, iff $X \times G_2$ is* (*left or right*) *shy in $G_1 \times G_2$.*

Lemma 3.4. *For $k \in \mathbb{N}$, let G_k be a unimodular Polish group, that is, not compact and let μ_k be a Haar measure on G_k. Let $(Y_k)_{k \in N}$ be such a family that $0 \le \mu_k(Y_k) < +\infty$. Then $\prod_{k \in \mathbb{N}} Y_k$ is shy generated by any two sided invariant generator of shy sets on the product-group $G = \prod_{k \in \mathbb{N}} G_k$.*

Proof. For $k \in \mathbb{N}$, the μ_k is not finite, and we can choose such a Borel set $Z_k \subset G_k$ that $2^k \mu_k(Y_k) < \mu_k(Z_k)$ and $\mu_k(Z_k) > 0$. For $k \in \mathbb{N}$, we set $\lambda_k = \frac{\mu_k}{\mu_k(Z_k)}$. Let μ be a two sided $\prod_{k \in \mathbb{N}} G_k$- invariant

Borel measure on G defined by the family $(\lambda_k)_{k\in\mathbb{N}}$ (see Lemma 3.2). Then we get

$$\mu(\prod_{k\in\mathbb{N}} Y_k) = \prod_{k\in\mathbb{N}} \lambda_k(Y_k) = \prod_{k\in\mathbb{N}} \frac{\mu_k(Y_k)}{\mu_k(Z_k)} \leq \prod_{k\in\mathbb{N}} \frac{1}{2^k} = 0.$$

By Corollary 2.2, μ is a generator of shy sets on the product-group G. $\square$

Now, it is not hard to prove the following lemma.

Lemma 3.5. *For $k \in \mathbb{N}$, let G_k be a locally compact Hausdorff topological group, that is, not compact and let μ_k be a left (or right) Haar measure on G_k. Let $(Y_k)_{k\in N}$ be such a family that $0 \leq \mu_k(Y_k) < +\infty$ for $k \in \mathbb{N}$. Then $\prod_{k\in\mathbb{N}} Y_k$ is left (or right)-shy set generated by any left (or right) invariant generator of left (or right) shy sets on the product-group $G = \prod_{k\in\mathbb{N}} G_k$.*

Lemma 3.6. *For $k \in \mathbb{N}$, let G_k be a locally compact Hausdorff topological group. Let K be a compact subset in $\prod_{k\in\mathbb{N}} G_k$. Then there exists a family $(Y_k)_{k\in\mathbb{N}}$ such that:*

(1) *Y_k is compact in G_k;*

(2) *$K \subset \prod_{k\in\mathbb{N}} Y_k$.*

We have the following corollaries of Lemmas 3.4-3.6.

Corollary 3.1. *For $k \in \mathbb{N}$, let G_k be a unimodular Polish group, that is, not compact. Then any compact subset (and hence any K_σ subset) of the product-group $\prod_{k\in\mathbb{N}} G_k$ is shy generated by any two sided invariant generator of shy sets on the entire group.*

Proof. Let K be any compact subset in $\prod_{k\in\mathbb{N}} G_k$. Let $(Y_k)_{k\in\mathbb{N}}$ be a family of subsets mentioned in Lemma 3.6. For $k \in \mathbb{N}$, let λ_k be a Haar measure defined on G_k. Since, λ_k is not finite, we claim that $0 \leq \lambda_k$

$(Y_k) < +\infty$ for $k \in \mathbb{N}$. By Lemma 3.4, we deduce that the set $\prod_{k\in\mathbb{N}} Y_k$ is shy in $\prod_{k\in\mathbb{N}} G_k$ generated by any two sided invariant generator μ of shy sets on $\prod_{k\in\mathbb{N}} G_k$. The latter relation implies that the set K, being a subset of the Borel shy set $\prod_{k\in\mathbb{N}} Y_k$, is shy generated by the two sided invariant generator of shy sets μ. □

Corollary 3.2. *For $k \in \mathbb{N}$, let G_k be an uncountable locally compact Hausdorff topological group, that is, not compact. Then any compact subset (and hence any K_σ subset) of $\prod_{k\in\mathbb{N}} G_k$ is left shy and right shy generated by any left invariant and right invariant generators of shy sets on the entire group, respectively.*

4. On a Certain Example of a Shy Set in the Product of Unimodular Polish Groups that are not Compact

The main result of the present section is formulated as follows.

Theorem 4.1. *For $k \in \mathbb{N}$, let $(G_k)_{k\in\mathbb{N}}$ be a family of unimodular Polish groups that are not compact. Let λ_k be a Haar measure on G_k. Let $\mathcal{H}$ be a class of Borel measurable rectangles of the form $\prod_{k\in\mathbb{N}} Y_k$ such that*

$$\mathrm{card}(\{k : 0 \leq \lambda_k(Y_k) < +\infty\} = \omega.$$

If a subset $X \subseteq \prod_{i\in\mathbb{N}} G_i$ is covered by the union of countable family of elements of $\mathcal{H}$, then X is shy in the product-group $G = \prod_{i\in\mathbb{N}} G_i$.

Proof. Firstly, let us show that every element of the class $\mathcal{H}$ is shy in G. Indeed, let $\prod_{k\in\mathbb{N}} Y_k \in \mathcal{H}$. We set $A = \{k : 0 \leq \lambda_k(Y_k) < +\infty\}$. Then by Lemma 3.4, we claim that the set $\prod_{k\in A} Y_k$ is shy in $\prod_{k\in A} G_k$. By Lemma 3.3, the set $\prod_{k\in A} Y_k \times \prod_{i\in\mathbb{N}\setminus A} G_i$ is shy in $\prod_{k\in\mathbb{N}} G_k$. Also, $\prod_{k\in\mathbb{N}} Y_k$ is shy as a subset of the shy set $\prod_{k\in A} Y_k \times \prod_{i\in\mathbb{N}\setminus A} G_i$.

Following [7] (see, Theorem 3, pp.32) the union of the countable family of elements of $\mathcal{H}$ is shy in G, and we claim that X is shy in G. □

In order to show that Theorem 4.1 extends Proposition 12(cf. [3], pp. 87) in the product of unimodular Polish groups that are not compact, we present the following example.

Example 4.1. Let $\mathbb{R}$ be an abelian Polish group of real numbers over the usual operation of addition. Then a space of real-valued sequences $\mathbb{R}^{\infty}$ equipped with usual product topology stands the product of a countable family of unimodular Polish groups that are not compact.

We set

$$(\forall k)(k \in \mathbb{N} \to \Delta_k = \sum_{i=1}^{\infty} \Delta_k^{(i)}),$$

where $\Delta_k^{(i)} = [3i;\ 3i + \frac{1}{2^i}]$ for $i \in \mathbb{N}$.

Let consider a Borel set $E = \prod_{k=1}^{\infty} \Delta_k$.

Let $(\mu_k)_{k \in \mathbb{N}}$ be a family of Lebesgue measures on $\mathbb{R}_k := \mathbb{R}$.

Clearly, $0 < l_k(\Delta_k) < +\infty$ for $k \in \mathbb{N}$. Hence, by Theorem 4.1, we claim that $\prod_{k \in \mathbb{N}} \Delta_k$ is shy in $\prod_{k \in \mathbb{N}} G_k$.

Now, let us show that E is not σ- compact.

Let $\mathcal{F}$ be a family of compact subsets in $\mathbb{R}^{\mathbb{N}}$ of the form $\prod_{k \in \mathbb{N}} F_k$, where F_k is compact in $\mathbb{R}_k$.

If we assume that E is covered by the union of countable family of compact sets in $\mathbb{R}^{\infty}$, then by Lemma 3.6, we can deduce that E is covered also by the union of any countable family $(R_j)_{j \in \mathbb{N}}$ of elements of $\mathcal{F}$, where $R_j = \prod_{k=1}^{\infty} F_k^{(j)}$ for $j \in \mathbb{N}$ and $F_k^{(j)}$ is compact in $\mathbb{R}_k$.

We have

$$E = \prod_{k\in N} \Delta_k = \prod_{k\in\mathbb{N}} (\sum_{i=1}^{\infty} \Delta_k^{(i)}).$$

For $j \in \mathbb{N}$, we choose $\Delta_j^{(i_j)}$ such that $F_j^{(j)} \cap \Delta_j^{(i_j)} = \emptyset$. Then $\prod_{j\in\mathbb{N}} \Delta_j^{(i_j)} \subset E$, and $(\forall j)(j \in \mathbb{N} \to R_j \cap \prod_{k=1}^{\infty} \Delta_k^{(i_k)} = \emptyset)$. We have obtained a required contradiction and thus, E is not σ- compact in $\mathbb{R}^{\infty}$.

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